ELSEVIER

Bull. Sci. math. 135 (2011) 400–419

BULLETIN DES SCIENCES MATHÉMATIQUES

www.elsevier.com/locate/bulsci

The relative Picard functor on schemes over a symmetric monoidal category

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Received 21 January 2011

Available online 4 February 2011

Abstract

We consider schemes $(X, \mathcal{O}_X)$ over an abelian closed symmetric monoidal category $(\mathbf{C}, \otimes, 1)$. Our aim is to extend a theorem of Kleiman on the relative Picard functor to schemes over $(\mathbf{C}, \otimes, 1)$. For this purpose, we also develop some basic theory on quasi-coherent modules on schemes $(X, \mathcal{O}_X)$ over $\mathbf{C}$.

MSC: 14K30; 18D10

Keywords: Picard functor; Symmetric monoidal categories

1. Introduction

The relative algebraic geometry over a symmetric monoidal category $(\mathbf{C}, \otimes, 1)$ has been studied widely in the literature (see Deligne [3], Hakim [6], Toën and Vaquie [13]). When $\mathbf{C} = k - Mod$, the category of modules over an ordinary commutative ring k, the relative algebraic geometry over $\mathbf{C}$ reduces to the usual algebraic geometry of schemes over $Spec(k)$.

The objective of this paper is to extend a theorem of Kleiman [10] on the relative Picard functor to schemes over an abelian closed symmetric monoidal category $(\mathbf{C}, \otimes, 1)$. For instance, $\mathbf{C}$ could be the category $k - Mod$ of modules over a commutative ring k, the category of sheaves of abelian groups over a topological space, the category of comodules over a flat Hopf algebroid (A, Γ) (see [8, Theorem 1.3]), the derived category of modules over a commutative ring k as well as chain complexes over all these categories (see [9, Definition 1.1.6] and [9, Proposition 9.2.1]).

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0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2011.02.001

In doing so, we further develop the notions from algebraic geometry in the abstract setting of symmetric monoidal categories.

More precisely, let $(\mathbf{C}, \otimes, 1)$ be a closed symmetric monoidal category containing limits and colimits and let $Comm(\mathbf{C})$ denote the category of commutative monoids in $\mathbf{C}$. Let $Aff_{\mathbf{C}} = Comm(\mathbf{C})^{op}$ be the category of affine schemes over $\mathbf{C}$. We consider the category $Sch_{\mathbf{C}}$ of schemes over $\mathbf{C}$ as defined by Toën and Vaquie [13]. Then, each scheme X over $\mathbf{C}$ can be associated to a "structure sheaf" $\mathcal{O}_X$ (see (2.3)). Further, the notion of Zariski coverings of schemes over $\mathbf{C}$ defined in [13] gives a Zariski topology on the category $Sch_{\mathbf{C}}$.

Given X in $Sch_{\mathbf{C}}$, we introduce the category $\mathcal{O}_X - QCoh$ of "quasi-coherent $\mathcal{O}_X$-modules" as well as a full subcategory $\mathcal{O}_X - Coh$ of $\mathcal{O}_X - QCoh$ consisting of "coherent $\mathcal{O}_X$-modules". Thereafter, if X is "bicomplete" (see Definition 2.3), we start by showing that $\mathcal{O}_X - Coh$ is a closed symmetric monoidal category. Further, for a morphism $f : X \to Y$ of bicomplete schemes, we introduce the functors $f_* : \mathcal{O}_X - QCoh \to \mathcal{O}_Y - QCoh$ and $f^* : \mathcal{O}_Y - QCoh \to \mathcal{O}_X - QCoh$ and describe their properties. Then, we refer to the abelian group of isomorphism classes of invertible, locally free coherent $\mathcal{O}_X$-modules (see Definitions 3.1 and 3.4) as the Picard group of X, denoted $Pic(X)$.

Let $h : X \to S$ be a morphism of schemes over $\mathbf{C}$. Then, we apply the results mentioned above to prove the following theorem, which generalizes a theorem of Kleiman [10, Theorem 2.5].

Theorem 1.1. *Let $h : X \to S$ be a morphism of bicomplete* (*see Definition* 2.3) *schemes over* $(\mathbf{C}, \otimes, 1)$. *Suppose that h is such that for any bicomplete S-scheme T, the pullback $h_T : X_T := X \times_S T \to T$ is bicomplete. Let $Pic_{X/S}$ denote the relative Picard functor*

$$Pic_{X/S} : (Sch_{\mathbf{C}}/S)^{op} \longrightarrow \mathbf{Ab} \qquad Pic_{X/S}(T) := Pic(X_T)/h_T^* Pic(T) \tag{1.1}$$

where $\mathbf{Ab}$ *denotes the category of abelian groups and by abuse of notation, $Sch_{\mathbf{C}}/S$ denotes the category of bicomplete schemes over S. Further, suppose that the natural morphism $\mathcal{O}_S \to h_*\mathcal{O}_X$ is universally an isomorphism, i.e., for any bicomplete S-scheme T, the natural morphism $\mathcal{O}_T \to h_{T*}\mathcal{O}_{X_T}$ is an isomorphism $\mathcal{O}_T \xrightarrow{\cong} h_{T*}\mathcal{O}_{X_T}$. Then, it follows that*:

(a) *The relative Picard functor $Pic_{X/S}$ defines a separated presheaf on $Sch_{\mathbf{C}}/S$.*
(b) *Additionally, suppose that the morphism $h : X \to S$ has a section $g : S \to X$ so that $hg = 1$. Then, the relative Picard functor $Pic_{X/S}$ defines a sheaf on $Sch_{\mathbf{C}}/S$.*

We mention here that in this paper, we shall restrict ourselves to the Zariski topology on schemes over $(\mathbf{C}, \otimes, 1)$, whereas in [10], etale and fppf topologies have also been considered. For similar work on the relative Picard functor in the context of stacks, we refer the reader to Brochard [1]. Throughout this paper, if $\mathscr{C}$ is any category, we will often write $X \in \mathscr{C}$ to mean that "X is an object of $\mathscr{C}$".

2. Quasi-coherent modules over a relative scheme

Let $(\mathbf{C}, \otimes, 1)$ be an abelian, closed symmetric monoidal category that contains all small limits and colimits. We recall here that in an abelian category, finite products and finite coproducts coincide and are often referred to as "biproducts". Consequently, given an object M in $\mathbf{C}$, for any integer $n \geqslant 1$, we shall use M^n to denote the biproduct of n-copies of M.

Let $Comm(\mathbf{C})$ denote the category of commutative monoid objects in $\mathbf{C}$. Since $\mathbf{C}$ is closed, for any commutative monoid $A \in Comm(\mathbf{C})$, the category $(A - Mod, \otimes_A, A)$ of A-modules is also a closed symmetric monoidal category (see, for instance, [15]), i.e., for any $M, N \in A - Mod$, there exists an object $\underline{Hom}_A(M, N) \in A - Mod$ such that the functor

$$P \mapsto Hom_A(M \otimes_A P, N) \quad \forall P \in A - Mod \tag{2.1}$$

is represented by $\underline{Hom}_A(M, N) \in A - Mod$.

Let $\mathit{Aff}_{\mathbf{C}} = Comm(C)^{op}$ be the category of affine schemes over $\mathbf{C}$. If A is an object of $Comm(\mathbf{C})$, we will often use $Spec(A)$ to denote the corresponding object in $\mathit{Aff}_{\mathbf{C}}$. Then, Toën and Vaquie [13, Définition 2.10] have introduced the notion of Zariski coverings in the category $\mathit{Aff}_{\mathbf{C}}$, determining a Grothendieck site that is also subcanonical, i.e. the representable presheaves on $\mathit{Aff}_{\mathbf{C}}$ are also sheaves. Accordingly, let $Sh(\mathit{Aff}_{\mathbf{C}})$ denote the category of sheaves of sets on $\mathit{Aff}_{\mathbf{C}}$. By abuse of notation, we will often denote the sheaf on $\mathit{Aff}_{\mathbf{C}}$ represented by an affine scheme $X \in \mathit{Aff}_{\mathbf{C}}$ also by X.

Further, in [13, Définition 2.12], Toën and Vaquie have introduced a suitable notion of Zariski open immersions in the category $Sh(\mathit{Aff}_{\mathbf{C}})$ that is stable under composition and base change. Together with this, we recall from [13, Définition 2.15], the following notion of a scheme over $\mathbf{C}$.

Definition 2.1. Let X be an object of $Sh(\mathit{Aff}_{\mathbf{C}})$. Then, X is a scheme over $(\mathbf{C}, \otimes, 1)$ if there exists a family $\{X_i\}_{i \in I}$ of affine schemes over $(\mathbf{C}, \otimes, 1)$ and a morphism

$$p : \coprod_{i \in I} X_i \longrightarrow X \tag{2.2}$$

satisfying the following conditions:

(a) The morphism p is an epimorphism in $Sh(\mathit{Aff}_{\mathbf{C}})$.
(b) For each $i \in I$, the morphism $X_i \to X$ is a Zariski open immersion in $Sh(\mathit{Aff}_{\mathbf{C}})$.

A collection of morphisms $(X_i \to X)_{i \in I}$ as in (2.2) will be referred to as an affine Zariski covering of X.

We mention here that the category of schemes over $\mathbf{C}$ defined above contains disjoint unions and fibre products. Let X be a scheme over $\mathbf{C}$ and let us consider the category $Sh(\mathit{Aff}_{\mathbf{C}})/X$. Let $\mathit{ZarAff}(X)$ denote the full subcategory of $Sh(\mathit{Aff}_{\mathbf{C}})/X$ whose objects are Zariski open immersions $U \to X$ with U an affine scheme over $\mathbf{C}$. When there is no danger of confusion, we will often refer to an object $U \to X$ in $\mathit{ZarAff}(X)$ simply as U. Then, (see [13, §2.4]) the scheme X defines a functor

$$\mathcal{O}_X : \mathit{ZarAff}(X)^{op} \longrightarrow Comm(\mathbf{C}) \tag{2.3}$$

that associates the object $Spec(A) = U \to X$ in $\mathit{ZarAff}(X)$ to $A \in Comm(\mathbf{C})$. Let $X, Y \in Sh(\mathit{Aff}_{\mathbf{C}})$ be schemes and let $f : X \to Y$ be a morphism of schemes over $\mathbf{C}$. Let $V \to Y$ be an object of $\mathit{ZarAff}(Y)$ and suppose that we have a cartesian square

$$\begin{array}{ccc} U & \xrightarrow{f'} & V \\ {\scriptstyle p'}\downarrow & & {\scriptstyle p}\downarrow \\ X & \xrightarrow{f} & Y \end{array} \tag{2.4}$$

Then, $p' : U \to X$ is also a Zariski open immersion and U is also a scheme over $\mathbf{C}$. We now introduce a functor

$$f_*\mathcal{O}_X : ZarAff(Y)^{op} \longrightarrow Comm(\mathbf{C}) \tag{2.5}$$

that associates the object $V \to Y$ of $ZarAff(Y)$ to the limit

$$f_*\mathcal{O}_X(V) := \lim_{W \in ZarAff(U)} \mathcal{O}_X(W) \tag{2.6}$$

For each $W \in ZarAff(U)$ as in (2.6), the composition $W \to U \xrightarrow{f'} V$ induces a morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(W)$ in $Comm(\mathbf{C})$. From (2.6), it follows that these morphisms together induce a unique morphism from $\mathcal{O}_Y(V)$ to the limit $f_*\mathcal{O}_X(V)$. Further, as $V \to Y$ varies over all of $ZarAff(Y)$, we have a natural transformation

$$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X \tag{2.7}$$

of functors from $ZarAff(Y)^{op}$ to $Comm(\mathbf{C})$. Henceforth, a scheme X over $\mathbf{C}$ will often be denoted by a pair $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the "structure sheaf" of X as in (2.3). A morphism $f : X \to Y$ of schemes will often be denoted by $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, where $f^\sharp$ is the natural transformation $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of functors from $ZarAff(Y)^{op}$ to $Comm(\mathbf{C})$ associated to f as in (2.7). The category of schemes over $\mathbf{C}$ will be denoted by $Sch_{\mathbf{C}}$.

Given a scheme $(X, \mathcal{O}_X)$ over $\mathbf{C}$, our next aim is to define a category of quasi-coherent $\mathcal{O}_X$-modules. For this, we consider the category $Mod_{\mathbf{C}}$ whose objects are pairs (A, M), where A is an object of $Comm(\mathbf{C})$ and M is an object of $A - Mod$. A morphism $(f, f_\sharp) : (A, M) \to (B, N)$ in the category $Mod_{\mathbf{C}}$ consists of a morphism $f : A \to B$ in $Comm(\mathbf{C})$ and a morphism $f_\sharp : B \otimes_A M \to N$ of B-modules.

Definition 2.2. Let $(X, \mathcal{O}_X)$ be a scheme over $\mathbf{C}$. By a quasi-coherent $\mathcal{O}_X$-module, we will mean a functor $\mathcal{M} : ZarAff(X)^{op} \to Mod_{\mathbf{C}}$ that satisfies the following two properties:

(a) If p denotes the obvious projection $p : Mod_{\mathbf{C}} \to Comm(\mathbf{C})$, we have $p \circ \mathcal{M} = \mathcal{O}_X$.
(b) For any morphism $u : U' \to U$ in $ZarAff(X)$, suppose that $\mathcal{M}(U) = (\mathcal{O}_X(U), M)$, $\mathcal{M}(U') = (\mathcal{O}_X(U'), M')$ and consider the induced morphism $\mathcal{M}(u) := (\mathcal{O}_X(u), \mathcal{O}_X(u)_\sharp) : (\mathcal{O}_X(U), M) \to (\mathcal{O}_X(U'), M')$ in $Mod_{\mathbf{C}}$. Then, the morphism $\mathcal{O}_X(u)_\sharp : \mathcal{O}_X(U') \otimes_{\mathcal{O}_X(U)} M \to M'$ is an isomorphism.

By a morphism of quasi-coherent $\mathcal{O}_X$-modules from $\mathcal{M}$ to $\mathcal{N}$, we will mean a natural transformation $T : \mathcal{M} \to \mathcal{N}$ of functors such that the induced transformation $p \circ T : \mathcal{O}_X = p \circ \mathcal{M} \to p \circ \mathcal{N} = \mathcal{O}_X$ of functors is the identity. The category of quasi-coherent $\mathcal{O}_X$-modules will be denoted by $\mathcal{O}_X - QCoh$. Given an object $\mathcal{M} \in \mathcal{O}_X - QCoh$, and any $U \in ZarAff(X)^{op}$, we will often use $\mathcal{M}(U)$ to denote only the $\mathcal{O}_X(U)$-module corresponding to $\mathcal{M}(U) \in Mod_{\mathbf{C}}$. Further, if $V \to X$ is any Zariski open immersion, we will denote by $\mathcal{M}|V$ the restriction of the functor $\mathcal{M} : ZarAff(X)^{op} \to Mod_{\mathbf{C}}$ to $ZarAff(V)^{op}$.

Since the category $\mathbf{C}$ is abelian, it contains a zero object 0, i.e., an object that is both an initial and a final object. For a commutative monoid A in $\mathbf{C}$, we will denote by $A - Coh$ the full subcategory of $A - Mod$ whose objects M may be described as finite colimits of the form

$$M = \operatorname{colim}\bigl(0 \longleftarrow A^m \xrightarrow{f} A^n\bigr) \quad m, n \in \mathbb{Z},\ m, n \geqslant 0 \tag{2.8}$$

Given a scheme $(X, \mathcal{O}_X)$ over $\mathbf{C}$, we denote by $\mathcal{O}_X - Coh$ the full subcategory of $\mathcal{O}_X - QCoh$ consisting of functors $\mathcal{M} : ZarAff(X)^{op} \to Mod_{\mathbf{C}}$ such that for any $U \in ZarAff(X)$, the object $\mathcal{M}(U) \in \mathcal{O}_X(U) - Mod$ lies in $\mathcal{O}_X(U) - Coh$. The objects of $\mathcal{O}_X - Coh$ will be referred to as coherent $\mathcal{O}_X$-modules. For the sake of convenience, the coherent $\mathcal{O}_X$-module defined by associating each object $U \to X$ in $ZarAff(X)^{op}$ to $(\mathcal{O}_X(U), \mathcal{O}_X(U)) \in Mod_{\mathbf{C}}$ will also be denoted by $\mathcal{O}_X$.

Definition 2.3. (See also Remark 2.9.) Let $(X, \mathcal{O}_X)$ be a scheme over $\mathbf{C}$. We will say that $(X, \mathcal{O}_X)$ is bicomplete, if it satisfies the following conditions:

(a) For any $U \in ZarAff(X)$, the category $\mathcal{O}_X(U) - Coh$ is both finitely complete and finitely cocomplete.
(b) Given any Zariski open immersion $U \to X$ and $\mathcal{M} \in \mathcal{O}_U - QCoh$ such that there exists a Zariski affine cover $(U_i \to U)_{i\in I}$ of U with each $\mathcal{M}(U_i) \in \mathcal{O}_U(U_i) - Coh$, then $\mathcal{M} \in \mathcal{O}_U - Coh$.
(c) Given any Zariski open immersion $V \to X$ along with an affine Zariski open cover $(V_j \to V)_{j\in J}$, there exists a finite subset $J' \subseteq J$ such that the collection $(V_j \to V)_{j\in J'}$ is a Zariski cover of V.

A commutative monoid object A in $\mathbf{C}$ will be said to be bicomplete if $Spec(A)$ is a bicomplete scheme in the sense of Definition 2.3.

Lemma 2.4. *Let A be a commutative monoid object in $\mathbf{C}$ that is also bicomplete. Let M, N be objects of $A - Coh$. Then, $M \otimes_A N$ and $\underline{Hom}_A(M, N)$ are also objects of $A - Coh$.*

Proof. Since $M, N \in A - Coh$, we can describe M and N as colimits as follows:

$$M = \mathrm{colim}\big(0 \longleftarrow A^m \xrightarrow{f} A^n\big) \qquad N = \mathrm{colim}\big(0 \longleftarrow A^k \xrightarrow{g} A^l\big) \tag{2.9}$$

Since $\mathbf{C}$ is closed, the functors $\_\_ \otimes_A A^l$, $\_\_ \otimes_A A^k$ and $M \otimes_A \_\_$ commute with colimits. Then, we have:

$$\begin{aligned} M \otimes_A A^l &= \mathrm{colim}\big(0 \longleftarrow A^{ml} \xrightarrow{f\otimes 1} A^{nl}\big) \\ M \otimes_A A^k &= \mathrm{colim}\big(0 \longleftarrow A^{mk} \xrightarrow{f\otimes 1} A^{nk}\big) \end{aligned} \tag{2.10}$$

and

$$M \otimes_A N = \mathrm{colim}\big(0 \longleftarrow M \otimes_A A^k \xrightarrow{1\otimes g} M \otimes_A A^l\big) \tag{2.11}$$

Since $A - Coh$ is finitely cocomplete, it follows from (2.10) and (2.11) that $M \otimes_A N$ is also an object of $A - Coh$.

Again, since $\mathbf{C}$ is closed, the functor $\underline{Hom}_A(\_\_, N)$ converts colimits into limits. It follows that

$$\begin{aligned} \underline{Hom}_A(M, N) &= \lim\big(0 \longrightarrow \underline{Hom}_A\big(A^m, N\big) \longleftarrow \underline{Hom}_A\big(A^n, N\big)\big) \\ &= \lim\big(0 \longrightarrow N^m \longleftarrow N^n\big) \end{aligned} \tag{2.12}$$

Since N^m and N^n are in $A - Coh$ and $A - Coh$ is closed under finite limits, it follows that $\underline{Hom}_A(M, N) \in A - Coh$. $\square$

Proposition 2.5. *Let $(X, \mathcal{O}_X)$ be a bicomplete scheme over $\mathbf{C}$. Then, the category $\mathcal{O}_X - Coh$ of coherent $\mathcal{O}_X$-modules is a closed symmetric monoidal category.*

Proof. We choose objects $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ in $\mathcal{O}_X - Coh$. Let $U \to X$ be an object of $ZarAff(X)$. Then, we define

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})(U) := \big(\mathcal{O}_X(U), \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)\big) \in Mod_{\mathbf{C}} \tag{2.13}$$

and

$$\underline{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P})(U) := \big(\mathcal{O}_X(U), \underline{Hom}_{\mathcal{O}_X(U)}\big(\mathcal{N}(U), \mathcal{P}(U)\big)\big) \in Mod_{\mathbf{C}} \tag{2.14}$$

From Lemma 2.4, it follows that both $\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$ and $\underline{Hom}_{\mathcal{O}_X(U)}(\mathcal{N}(U), \mathcal{P}(U))$ are in $\mathcal{O}_X(U) - Coh$.

If $u : V \to U$ is a morphism in $ZarAff(X)$, we note that

$$\begin{aligned}
&(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \\
&\quad \cong \big(\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)\big) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \\
&\quad \cong \big(\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)\big) \otimes_{\mathcal{O}_X(V)} \big(\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)\big) \\
&\quad \cong \mathcal{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{N}(V) = (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})(V)
\end{aligned} \tag{2.15}$$

and hence $(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \in \mathcal{O}_X - Coh$. Moreover, since $\mathcal{N}(U)$ is an object of $\mathcal{O}_X(U) - Coh$, there exist non-negative integers m, n such that

$$\begin{aligned}
\mathcal{N}(U) &= \operatorname{colim}\big(0 \longleftarrow \mathcal{O}_X(U)^m \longrightarrow \mathcal{O}_X(U)^n\big) \\
\mathcal{N}(V) &= \operatorname{colim}\big(0 \longleftarrow \mathcal{O}_X(V)^m \longrightarrow \mathcal{O}_X(V)^n\big)
\end{aligned} \tag{2.16}$$

where the latter equality in (2.16) follows by applying $\_\_ \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$ to the former. Applying $\underline{Hom}_{\mathcal{O}_X(U)}(\_\_, \mathcal{P}(U))$ to the first equality in (2.16), we get

$$\underline{Hom}_{\mathcal{O}_X(U)}\big(\mathcal{N}(U), \mathcal{P}(U)\big) = \lim\big(0 \longrightarrow \mathcal{P}(U)^m \longleftarrow \mathcal{P}(U)^n\big) \tag{2.17}$$

The morphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is induced by a Zariski open immersion and hence is flat (see [13, Définition 2.12]). Therefore, the functor $\_\_ \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$ commutes with finite limits. Hence, it follows from (2.17) that

$$\underline{Hom}_{\mathcal{O}_X(U)}\big(\mathcal{N}(U), \mathcal{P}(U)\big) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) = \lim\big(0 \longrightarrow \mathcal{P}(V)^m \longleftarrow \mathcal{P}(V)^n\big) \tag{2.18}$$

Applying the functor $\underline{Hom}_{\mathcal{O}_X(V)}(\_\_, \mathcal{P}(V))$ to the latter equality in (2.16), we get

$$\underline{Hom}_{\mathcal{O}_X(V)}\big(\mathcal{N}(V), \mathcal{P}(V)\big) = \lim\big(0 \longrightarrow \mathcal{P}(V)^m \longleftarrow \mathcal{P}(V)^n\big) \tag{2.19}$$

Comparing (2.18) and (2.19), we have

$$\underline{Hom}_{\mathcal{O}_X(U)}\big(\mathcal{N}(U), \mathcal{P}(U)\big) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \cong \underline{Hom}_{\mathcal{O}_X(V)}\big(\mathcal{N}(V), \mathcal{P}(V)\big) \tag{2.20}$$

and hence $\underline{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P}) \in \mathcal{O}_X - Coh$. Finally, since the isomorphisms

$$\begin{aligned}
&Hom_{\mathcal{O}_X(U)}\big(\mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U), \mathcal{P}(U)\big) \\
&\quad \cong Hom_{\mathcal{O}_X(U)}\big(\mathcal{M}(U), \underline{Hom}_{\mathcal{O}_X(U)}\big(\mathcal{N}(U), \mathcal{P}(U)\big)\big)
\end{aligned} \tag{2.21}$$

are natural and functorial, we have natural isomorphisms:

$$Hom_{\mathcal{O}_X - Coh}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}, \mathcal{P}) \cong Hom_{\mathcal{O}_X - Coh}\big(\mathcal{M}, \underline{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{P})\big) \tag{2.22}$$

This proves that $\mathcal{O}_X - Coh$ is a closed symmetric monoidal category. $\square$

Let $(X, \mathcal{O}_X) \in Sch_{\mathbf{C}}$ and let $U \to X$ be an object of $ZarAff(X)$. Then, we will denote by $Cov(U \to X)$ the set of all families $\mathcal{U} = (U_k \to U)_{k \in K}$ satisfying the following two conditions:

(C1) Each U_k, $k \in K$ is affine, each morphism $U_k \to U$, $k \in K$ is a Zariski open immersion and there exists a subset $K' \subseteq K$ such that $(U_{k'} \to U)_{k' \in K'}$ forms an affine Zariski cover of U.

(C2) For any pair $k_1', k_2' \in K' \subseteq K$, there exists a subset $K'(k_1', k_2') \subseteq K$ such that each map $(U_{k''} \to U)$, $k'' \in K'(k_1', k_2')$ factors through the fibre product $U_{k_1'} \times_U U_{k_2'}$ and the family $(U_{k''} \to U_{k_1'} \times_U U_{k_2'})$, $k'' \in K'(k_1', k_2')$ forms an affine Zariski cover of $U_{k_1'} \times_U U_{k_2'}$. Further, $K = K' \cup \bigcup_{(k_1', k_2') \in K' \times K'} K'(k_1', k_2')$.

We will now construct a pushforward on quasi-coherent $\mathcal{O}_X$-modules. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of (not necessarily bicomplete) schemes over $\mathbf{C}$. We choose some $\mathcal{M} \in \mathcal{O}_X - QCoh$. Let $V \to Y$ be an object of $ZarAff(Y)$ and suppose that we have a cartesian square:

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "p'"'] & V \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.23}$$

We choose any family $\mathcal{U} = (U_k \to U)_{k \in K}$ in $Cov(U \to X)$. Then, in the notation of conditions (C1) and (C2) above, we set

$$\begin{aligned} f_*\mathcal{M}(V) &:= \lim\Bigg(\prod_{k' \in K' \subseteq K} \mathcal{M}(U_{k'}) \rightrightarrows \prod_{\substack{k'' \in K'(k_1', k_2') \\ (k_1', k_2') \in K' \times K'}} \mathcal{M}(U_{k''}) \Bigg) \\ &= \lim_{W \in ZarAff(U)} \mathcal{M}(W) \end{aligned} \tag{2.24}$$

Since the families in $Cov(U \to X)$ form an inverse system and $\mathcal{M} \in \mathcal{O}_X - QCoh$, it follows that $f_*\mathcal{M}(V)$ as defined in (2.24) does not depend on the choice of $\mathcal{U}$ in $Cov(U \to X)$ and that $f_*\mathcal{M}(V)$ is an $f_*\mathcal{O}_X(V)$-module, where $f_*\mathcal{O}_X(V)$ is as in (2.6). Further, we have a transformation $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of functors, inducing a morphism $\mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V)$ in $Comm(\mathbf{C})$. Hence, $f_*\mathcal{M}(V)$ becomes a module over $\mathcal{O}_Y(V)$ by "restriction of scalars".

Proposition 2.6. *Let $(Y, \mathcal{O}_Y)$ be a bicomplete scheme over $\mathbf{C}$ and let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism in $Sch_{\mathbf{C}}$.*

(1) *Suppose that X is bicomplete or that f satisfies the following two conditions*:
  (a) *The morphism f is quasi-compact, i.e., for any $V \in ZarAff(Y)$, $U = V \times_Y X$, any Zariski affine open cover $(U_j \to U)_{j \in J}$ of U has a finite subcover.*
  (b) *For any $V \in ZarAff(Y)$, $U = V \times_Y X$ and U_1, $U_2 \in ZarAff(U)$, the fibre product $U_1 \times_U U_2$ lies in $ZarAff(U)$.*

Then, the association of $V \in ZarAff(Y)$ to $f_*\mathcal{M}(V)$ as in (2.24) *defines a functor $f_* : \mathcal{O}_X - QCoh \to \mathcal{O}_Y - QCoh$.*

(2) *Suppose that X is bicomplete and f is such that for any $V \in ZarAff(Y)$, $U = V \times_Y X$, we have $U \in ZarAff(X)$ and $\mathcal{O}_X(U)$ is an object of $\mathcal{O}_Y(V) - Coh$. Then, the functor $f_* : \mathcal{O}_X - QCoh \to \mathcal{O}_Y - QCoh$ restricts to $f_* : \mathcal{O}_X - Coh \to \mathcal{O}_Y - Coh$.*

Proof. (1) We will maintain all the notation from (2.23) and (2.24). Let $\mathcal{M} \in \mathcal{O}_X - QCoh$ and for any $V \in ZarAff(Y)$, let $U := V \times_Y X$ and let $f_*\mathcal{M}(V)$ be as defined in (2.24).

If $v : V' \to V$ is a morphism in $ZarAff(Y)$, $v : V' \to V$ is a Zariski open immersion and hence $\mathcal{O}_Y(V')$ is a flat $\mathcal{O}_Y(V)$ module. Hence, the functor $\_\_ \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y(V')$ commutes with finite limits. Let $u : U' := U \times_V V' \to U$ be the morphism induced by $v : V' \to V$. For any $\mathcal{U} = (U_k \to U)_{k\in K}$ in $Cov(U \to X)$, we define the family $\mathcal{U}' := (U'_k \to U')_{k\in K}$ by setting $U'_k := U_k \times_U U' \cong U_k \times_V V'$. Since V, V' and U_k are all affine, we note that each U'_k, $k \in K$ is affine and hence $\mathcal{U}' \in Cov(U' \to X)$. For any $k \in K$, we now have

$$\begin{aligned}\mathcal{M}(U_k) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y\big(V'\big) &\cong \mathcal{M}(U_k) \otimes_{\mathcal{O}_X(U_k)} \big(\mathcal{O}_X(U_k) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y\big(V'\big)\big) \\ &\cong \mathcal{M}(U_k) \otimes_{\mathcal{O}_X(U_k)} \mathcal{O}_X\big(U'_k\big) \cong \mathcal{M}\big(U'_k\big) \end{aligned} \tag{2.25}$$

If f satisfies condition (a), i.e. f is quasi-compact, we may assume that the subset $K' \subseteq K$ such that $(U_k \to U)_{k\in K'}$ forms a cover of U is finite. Further, if f satisfies condition (b), we know that each $U_{k'_1} \times_U U_{k'_2} \in ZarAff(U)$ for any $k'_1, k'_2 \in K'$ and hence we may assume that each $K'(k'_1, k'_2)$ is a singleton set.

On the other hand, if X is bicomplete, it follows directly from Definition 2.3 that the subset $K' \subseteq K$ as well as the subsets $K'(k'_1, k'_2) \subseteq K$, $k'_1, k'_2 \in K'$ are all finite.

Now, since $\mathcal{O}_Y(V')$ is a flat $\mathcal{O}_Y(V)$-module, it now follows that

$$\begin{aligned} & f_*\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y\big(V'\big) \\ &\quad \cong \lim\Big(\prod_{k'\in K'\subseteq K} \mathcal{M}(U_{k'}) \rightrightarrows \prod_{\substack{k''\in K'(k'_1,k'_2)\\ (k'_1,k'_2)\in K'\times K'}} \mathcal{M}(U_{k''})\Big) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y\big(V'\big) \\ &\quad \cong \lim\Big(\prod_{k'\in K'\subseteq K} \mathcal{M}(U_{k'}) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y\big(V'\big) \rightrightarrows \prod_{\substack{k''\in K'(k'_1,k'_2)\\ (k'_1,k'_2)\in K'\times K'}} \mathcal{M}(U_{k''}) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y\big(V'\big)\Big) \\ &\quad \cong \lim\Big(\prod_{k'\in K'\subseteq K} \mathcal{M}\big(U'_{k'}\big) \rightrightarrows \prod_{\substack{k''\in K'(k'_1,k'_2)\\ (k'_1,k'_2)\in K'\times K'}} \mathcal{M}\big(U'_{k''}\big)\Big) \\ &\quad \cong f_*\mathcal{M}\big(V'\big) \end{aligned}$$

Hence, we have $f_*\mathcal{M} \in \mathcal{O}_Y - QCoh$.

(2) In particular, suppose that $\mathcal{M} \in \mathcal{O}_X - Coh$, X is bicomplete and f satisfies the conditions in (2). In the notation above, it follows that for any $V \in ZarAff(Y)$ we have $U = V \times_Y X \in ZarAff(X)$. Then, $1 : U \to U$ is an element of $Cov(U \to X)$ and accordingly, we have $f_*\mathcal{M}(V) = \mathcal{M}(U)$. By assumption, we also know that $\mathcal{O}_X(U) \in \mathcal{O}_Y(V) - Coh$. Since $\mathcal{M}(U) \in \mathcal{O}_X(U) - Coh$, there exist integers $m, n \in \mathbb{Z}$ such that

$$\mathcal{M}(U) = \operatorname{colim}\big(0 \longleftarrow \mathcal{O}_X(U)^m \longrightarrow \mathcal{O}_X(U)^n\big) \tag{2.26}$$

The category $\mathcal{O}_Y(V) - Coh$ being closed under finite colimits (since Y is bicomplete), it follows that $f_*\mathcal{M}(V) = \mathcal{M}(U) \in \mathcal{O}_Y(V) - Coh$. Since X is bicomplete, it follows from part (1) that $f_*\mathcal{M} \in \mathcal{O}_Y - QCoh$. Hence, $f_*\mathcal{M} \in \mathcal{O}_Y - Coh$. $\square$

Remark 2.7. We note that condition (b) in part (1) of Proposition 2.6 can be compared to the properties of separated morphisms (see [7, Ex. II.4.3]) in usual algebraic geometry. Then, one may compare (1) of Proposition 2.6 to [7, Proposition II.5.8(c)].

Proposition 2.8. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of bicomplete schemes in $Sch_{\mathbf{C}}$. Then, there exists a pullback functor $f^* : \mathcal{O}_Y - QCoh \to \mathcal{O}_X - QCoh$ that restricts to a functor $f^* : \mathcal{O}_Y - Coh \to \mathcal{O}_X - Coh$. Further, the functor $f^* : \mathcal{O}_Y - QCoh \to \mathcal{O}_X - QCoh$ satisfies $f^*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}) \cong f^*\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{N}$ for $\mathcal{M}, \mathcal{N} \in \mathcal{O}_Y - QCoh$.*

Proof. Let $\mathcal{M} \in \mathcal{O}_Y - QCoh$. Let $V \to Y$ be an object of $ZarAff(Y)$. We form the cartesian square:

$$\begin{array}{ccc} U & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & Y \end{array} \tag{2.27}$$

Then, for any Zariski open immersion $U' \to U$ with U' affine, we have a morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U')$ in $Comm(\mathbf{C})$ and we set $(f^*\mathcal{M}_V)(U') := \mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U')$. Then, $f^*\mathcal{M}_V \in \mathcal{O}_U - QCoh$.

Suppose that $v : V'' \to V$ is a morphism in $ZarAff(Y)$ and let $u : U'' := U \times_V V'' \to U$ be the morphism induced by v. Let $U''' \to U''$ be a Zariski open immersion with U''' affine. Then, we note that

$$\begin{aligned} \big(f^*\mathcal{M}_V\big)\big(U'''\big) &= \mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X\big(U'''\big) \\ &\cong \mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y\big(V''\big) \otimes_{\mathcal{O}_Y(V'')} \mathcal{O}_X\big(U'''\big) \\ &\cong \mathcal{M}\big(V''\big) \otimes_{\mathcal{O}_Y(V'')} \mathcal{O}_X\big(U'''\big) \\ &\cong \big(f^*\mathcal{M}_{V''}\big)\big(U'''\big) \end{aligned} \tag{2.28}$$

From the natural isomorphisms in (2.28), it follows that as $V \to Y$ varies over $ZarAff(Y)$, the $f^*\mathcal{M}_V \in \mathcal{O}_U - QCoh$ (with $U := X \times_Y V$) can be glued together to define an object $f^*\mathcal{M} \in \mathcal{O}_X - QCoh$ which may be described as follows: we consider an affine Zariski cover $(V_i \to Y)_{i \in I}$ of Y and let $U_i := X \times_Y V_i$. Let $W \to X$ be an arbitrary object in $ZarAff(X)$. Then we can choose a finite family $\mathcal{W} = (W_k \to W)_{k \in K}$ in $Cov(W \to X)$ such that for each $k \in K$, there exists $t(k) \in I$ such that the composition $(W_k \to W \to X)$ factors through $U_{t(k)} \to X$. Then, $f^*\mathcal{M}(W)$ may be described as the limit:

$$f^*\mathcal{M}(W) = \lim_{W_k \in \mathcal{W}} f^*\mathcal{M}_{V_{t(k)}}(W_k) \tag{2.29}$$

In particular, suppose that $\mathcal{M} \in \mathcal{O}_Y - Coh$. From the above, it is clear that if $W' \to X$ is any object of $ZarAff(X)$ factoring through some U_i, we have $f^*\mathcal{M}(W') = \mathcal{M}(V_i) \otimes_{\mathcal{O}_Y(V_i)} \mathcal{O}_X(W') \in \mathcal{O}_X(W') - Coh$. Since X is bicomplete, it follows from Definition 2.3 that $f^*\mathcal{M} \in \mathcal{O}_X - Coh$.

It remains to show that for any $\mathcal{M}, \mathcal{N} \in \mathcal{O}_Y - QCoh$, we have $f^*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}) \cong f^*\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{N}$. In the notation above, it suffices to check this for an affine Zariski cover $(U_{ij} \to U_i)_{j \in J_i}$ of each U_i. Then, for any $i \in I$ and $j \in J_i$, we have

$$
\begin{aligned}
&\big(f^*\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{N}\big)(U_{ij}) \\
&\quad = f^*\mathcal{M}(U_{ij}) \otimes_{\mathcal{O}_X(U_{ij})} f^*\mathcal{N}(U_{ij}) \\
&\quad = \big(\mathcal{M}(V_i) \otimes_{\mathcal{O}_Y(V_i)} \mathcal{O}_X(U_{ij})\big) \otimes_{\mathcal{O}_X(U_{ij})} \big(\mathcal{N}(V_i) \otimes_{\mathcal{O}_Y(V_i)} \mathcal{O}_X(U_{ij})\big) \\
&\quad \cong \big(\mathcal{M}(V_i) \otimes_{\mathcal{O}_Y(V_i)} \mathcal{N}(V_i)\big) \otimes_{\mathcal{O}_Y(V_i)} \mathcal{O}_X(U_{ij}) \\
&\quad = (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(V_i) \otimes_{\mathcal{O}_Y(V_i)} \mathcal{O}_X(U_{ij}) \\
&\quad = f^*(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U_{ij}) \qquad \square
\end{aligned}
\tag{2.30}
$$

Remark 2.9. We may compare Proposition 2.8 above to [7, Proposition II.5.8(a), (b)]. This suggests that the notion of bicomplete schemes over **C** can be compared to some features of usual Noetherian schemes.

Let $(\mathbf{C}', \otimes, 1)$ be a closed symmetric monoidal category and for each object $Y \in \mathbf{C}'$, we define the dual DY to be the internal *Hom* object $DY := \underline{Hom}(Y, 1)$. Then, following [11, III.1.1], [12], Y is said to be dualizable if the natural morphism

$$DY \otimes Y = \underline{Hom}(Y, 1) \otimes Y \to \underline{Hom}(Y, Y) \tag{2.31}$$

is an isomorphism. Dualizable objects have also been referred to as "strongly dualizable" or "finite" objects in the literature (see also similar earlier notions in [2] and [4]). We mention here that when $\mathbf{C}'$ is the category $R - Mod$ of modules over a commutative ring R, an object $M \in R - Mod$ is dualizable if and only if it is a finitely generated and projective R-module.

Let $(X, \mathcal{O}_X)$ be a bicomplete scheme over **C**. From Proposition 2.5, it follows that $\mathcal{O}_X - Coh$ is a closed symmetric monoidal category. In particular, we will say that an object $\mathcal{M}$ in $\mathcal{O}_X - Coh$ is dualizable if the natural morphism

$$D\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M} = \underline{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \to \underline{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}) \tag{2.32}$$

is an isomorphism, where $D\mathcal{M} := \underline{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$. The object $D\mathcal{M}$ will be referred to as the dual of $\mathcal{M}$. We will denote by $\mathscr{D}(\mathcal{O}_X - Coh)$ the full subcategory of dualizable objects in $\mathcal{O}_X - Coh$.

Lemma 2.10. *Let $(\mathbf{C}', \otimes, 1)$ be a closed symmetric monoidal category. Then, if $Y \in \mathbf{C}'$ is dualizable, the functor $\_\_ \otimes Y$ commutes with limits in $\mathbf{C}'$. If $(X, \mathcal{O}_X)$ is a bicomplete scheme over* **C** *and $\mathcal{P} \in \mathscr{D}(\mathcal{O}_X - Coh)$, then for any object $V \to X$ of ZarAff(X), $\mathcal{P}(V)$ is a flat $\mathcal{O}_X(V)$-module.*

Proof. Let Z_i, $i \in I$ be a system of objects of $\mathbf{C}'$ and let $Z = \lim_{i \in I} Z_i$. Let DY denote the dual of $Y \in \mathbf{C}'$. Since Y is dualizable, we know that $DDY \cong Y$ (see [12, Proposition 2.7]). Then, for any object $W \in \mathbf{C}'$, we have natural isomorphisms (see [12, Theorem 2.6])

$$
\begin{aligned}
Hom(W, Z \otimes Y) &\cong Hom(W \otimes DY, Z) \\
&= Hom\Big(W \otimes DY, \lim_{i \in I} Z_i\Big) \\
&\cong \lim_{i \in I} Hom(W \otimes DY, Z_i) \\
&\cong \lim_{i \in I} Hom(W, Z_i \otimes DDY) \\
&\cong \lim_{i \in I} Hom(W, Z_i \otimes Y)
\end{aligned}
\tag{2.33}
$$

From Yoneda Lemma, it now follows that

$$Z \otimes Y = \Big(\lim_{i\in I} Z_i\Big) \otimes Y \cong \lim_{i\in I}(Z_i \otimes Y) \tag{2.34}$$

In particular, we know from Proposition 2.5 that for any scheme $(X, \mathcal{O}_X)$, $\mathcal{O}_X - Coh$ is a closed symmetric monoidal category. Let $\mathcal{P} \in \mathscr{D}(\mathcal{O}_X - Coh)$. Then, by definition, we have a natural isomorphism in $\mathcal{O}_X - Coh$

$$\underline{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{P} \cong \underline{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P}) \tag{2.35}$$

Hence, for any object $V \to X$ in $ZarAff(X)$, we get

$$\begin{aligned}\underline{Hom}_{\mathcal{O}_X(V)}\big(\mathcal{P}(V), \mathcal{O}_X(V)\big) \otimes_{\mathcal{O}_X(V)} \mathcal{P}(V) &= \big(\underline{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{P}\big)(V)\\ &\cong \underline{Hom}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P})(V)\\ &= \underline{Hom}_{\mathcal{O}_X(V)}\big(\mathcal{P}(V), \mathcal{P}(V)\big)\end{aligned} \tag{2.36}$$

From (2.36), it follows that $\mathcal{P}(V)$ is also a dualizable object of the closed symmetric monoidal category $\mathcal{O}_X(V) - Mod$. From (2.34), it now follows that the functor $\_\_ \otimes_{\mathcal{O}_X(V)} \mathcal{P}(V)$ commutes with finite limits in $\mathcal{O}_X(V) - Mod$ and hence $\mathcal{P}(V)$ is a flat $\mathcal{O}_X(V)$-module. □

Proposition 2.11. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of bicomplete schemes in $Sch_{\mathbf{C}}$. Then, if $\mathcal{M} \in \mathcal{O}_X - Coh$ and $\mathcal{P} \in \mathscr{D}(\mathcal{O}_Y - Coh)$, we have natural isomorphisms:*

$$f_*\big(\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{P}\big) \cong f_*(\mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{P} \tag{2.37}$$

Proof. Since X and Y are bicomplete, it follows from Proposition 2.6 that both $f_*(\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{P})$ and $f_*(\mathcal{M})$ lie in $\mathcal{O}_Y - QCoh$. Let $V \to Y$ be an object of $ZarAff(Y)$ and let us form the cartesian square

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \tag{2.38}$$

Let $\mathcal{U} = (U_k \to U)_{k\in K}$ be an element of $Cov(U \to X)$. Since X is bicomplete, we may assume K is finite. For each $k \in K$, we have:

$$\begin{aligned}\mathcal{M}(U_k) \otimes_{\mathcal{O}_X(U_k)} f^*\mathcal{P}(U_k) &\cong \mathcal{M}(U_k) \otimes_{\mathcal{O}_X(U_k)} \big(\mathcal{O}_X(U_k) \otimes_{\mathcal{O}_Y(V)} \mathcal{P}(V)\big)\\ &\cong \mathcal{M}(U_k) \otimes_{\mathcal{O}_Y(V)} \mathcal{P}(V)\end{aligned} \tag{2.39}$$

Since $\mathcal{P} \in \mathscr{D}(\mathcal{O}_Y - Coh)$, from Lemma 2.10, we know that $\mathcal{P}(V)$ is a flat $\mathcal{O}_Y(V)$-module. Hence, from (2.39) and the definitions in (2.24), it follows that (in the notation of conditions (C1) and (C2)):

$$\begin{aligned}&f_*\big(\mathcal{M} \otimes_{\mathcal{O}_X} f^*\mathcal{P}\big)(V)\\ &\quad\cong \lim\Bigg(\prod_{k'\in K'\subseteq K} \mathcal{M}(U_{k'}) \otimes_{\mathcal{O}_X(U_{k'})} f^*\mathcal{P}(U_{k'})\\ &\qquad\rightrightarrows \prod_{\substack{k''\in K'(k'_1,k'_2)\\ (k'_1,k'_2)\in K'\times K'}} \mathcal{M}(U_{k''}) \otimes_{\mathcal{O}_X(U_{k''})} f^*\mathcal{P}(U_{k''})\Bigg)\end{aligned}$$

$$\cong \lim\Bigg(\prod_{k'\in K'\subseteq K} \mathcal{M}(U_{k'}) \otimes_{\mathcal{O}_Y(V)} \mathcal{P}(V) \rightrightarrows \prod_{\substack{k''\in K'(k_1',k_2')\\ (k_1',k_2')\in K'\times K'}} \mathcal{M}(U_{k''}) \otimes_{\mathcal{O}_Y(V)} \mathcal{P}(V)\Bigg)$$

$$\cong \lim\Bigg(\prod_{k'\in K'\subseteq K} \mathcal{M}(U_{k'}) \rightrightarrows \prod_{\substack{k''\in K'(k_1',k_2')\\ (k_1',k_2')\in K'\times K'}} \mathcal{M}(U_{k''})\Bigg) \otimes_{\mathcal{O}_Y(V)} \mathcal{P}(V)$$

$$\cong f_*(\mathcal{M})(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{P}(V)$$

$$\cong \big(f_*(\mathcal{M}) \otimes_{\mathcal{O}_Y} \mathcal{P}\big)(V) \qquad \square \tag{2.40}$$

Proposition 2.12. *Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of bicomplete schemes in $Sch_{\mathbf{C}}$. Then, (f^*, f_*) forms a pair of adjoint functors from $\mathcal{O}_Y - QCoh$ to $\mathcal{O}_X - QCoh$.*

Proof. Let $\mathcal{M} \in \mathcal{O}_Y - QCoh$ and $\mathcal{N} \in \mathcal{O}_X - QCoh$. Let $V \to Y$ be an object of $ZarAff(Y)$ and let U be defined by the cartesian square

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array} \tag{2.41}$$

Let $(U_k \to U)_{k\in K}$ be chosen from $Cov(U \to X)$. Suppose that we have a morphism $g : \mathcal{M} \to f_*\mathcal{N}$ in $\mathcal{O}_Y - QCoh$. From the definitions in (2.24), it is clear that

$$f_*\mathcal{N}(V) = \lim_{k\in K} \mathcal{N}(U_k) \tag{2.42}$$

and hence g induces morphisms $g_k(V) : \mathcal{M}(V) \to \mathcal{N}(U_k)$, $k \in K$ of $\mathcal{O}_Y(V)$-modules. We note that:

$$\begin{aligned} Hom_{\mathcal{O}_Y(V)}\big(\mathcal{M}(V), \mathcal{N}(U_k)\big) &\cong Hom_{\mathcal{O}_X(U_k)}\big(\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U_k), \mathcal{N}(U_k)\big) \\ &= Hom_{\mathcal{O}_X(U_k)}\big(f^*\mathcal{M}(U_k), \mathcal{N}(U_k)\big) \end{aligned} \tag{2.43}$$

and let $g_k'(V) \in Hom_{\mathcal{O}_X(U_k)}(f^*\mathcal{M}(U_k), \mathcal{N}(U_k))$ denote the image of the morphism $g_k(V) \in Hom_{\mathcal{O}_Y(V)}(\mathcal{M}(V), \mathcal{N}(U_k))$ under the isomorphism in (2.43). Hence, if $V \to Y$ varies over all of $ZarAff(Y)$, the morphisms $g_k'(V) : f^*\mathcal{M}(U_k) \to \mathcal{N}(U_k)$ together define a morphism $g' : f^*\mathcal{M} \to \mathcal{N}$ in $\mathcal{O}_X - QCoh$.

Conversely, suppose that we have a morphism $h : f^*\mathcal{M} \to \mathcal{N}$ in $\mathcal{O}_X - QCoh$. Then, in the same notation as in (2.41) and (2.43) we have:

$$\begin{aligned} Hom_{\mathcal{O}_X(U_k)}\big(f^*\mathcal{M}(U_k), \mathcal{N}(U_k)\big) &= Hom_{\mathcal{O}_X(U_k)}\big(\mathcal{M}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U_k), \mathcal{N}(U_k)\big) \\ &\cong Hom_{\mathcal{O}_Y(V)}\big(\mathcal{M}(V), \mathcal{N}(U_k)\big) \end{aligned} \tag{2.44}$$

Hence, the morphism $h : f^*\mathcal{M} \to \mathcal{N}$ in $\mathcal{O}_X - QCoh$ induces a morphism $h_k(V) : \mathcal{M}(V) \to \mathcal{N}(U_k)$ in $\mathcal{O}_Y(V) - Mod$ for each $k \in K$. The morphisms $h_k(V)$, $k \in K$ together induce a unique morphism $h'(V) : \mathcal{M}(V) \to f_*\mathcal{N}(V) = \lim_{k\in K} \mathcal{N}(U_k)$ of $\mathcal{O}_Y(V)$-modules. As $V \to Y$ varies over all of $ZarAff(Y)$, it follows that we actually have a morphism $h' : \mathcal{M} \to f_*\mathcal{N}$ in $\mathcal{O}_Y - QCoh$. It is clear that these two associations are inverse to each other and hence we have a natural isomorphism

$$Hom_{\mathcal{O}_X - QCoh}\big(f^*\mathcal{M}, \mathcal{N}\big) \cong Hom_{\mathcal{O}_Y - QCoh}(\mathcal{M}, f_*\mathcal{N}) \qquad \square \tag{2.45}$$

3. The relative Picard functor

Let $(X, \mathcal{O}_X)$ be a bicomplete scheme in $Sch_{\mathbf{C}}$. In the previous section, we have developed the theory we need to study the relative Picard functor in the context of schemes over $\mathbf{C}$. More precisely, throughout this section, we let

$$h : (X, \mathcal{O}_X) \longrightarrow (S, \mathcal{O}_S) \tag{3.1}$$

denote a morphism of bicomplete schemes in $Sch_{\mathbf{C}}$. We will always suppose that the morphism $h : X \to S$ in (3.1) has the following property: given any bicomplete scheme T over S, the pullback $X_T := X \times_S T$ is also a bicomplete scheme. By abuse of notation, we will always let $Sch_{\mathbf{C}}/S$ denote the category of bicomplete schemes over S. We shall then define a relative Picard functor (see (3.16)):

$$Pic_{X/S} : (Sch_{\mathbf{C}}/S)^{op} \longrightarrow \mathbf{Ab} \tag{3.2}$$

where $\mathbf{Ab}$ denotes the category of abelian groups. The category $Sch_{\mathbf{C}}/S$ carries a Zariski topology induced from the Zariski topology on $Sch_{\mathbf{C}}$ and we let $\mathscr{P}ic_{X/S}$ denote the sheafification of the presheaf $Pic_{X/S}$ on $Sch_{\mathbf{C}}/S$. Then, we shall show that the argument of Kleiman [10, Theorem 2.5] can be generalized to give sufficient conditions under which the natural morphism

$$Pic_{X/S}(T) \longrightarrow \mathscr{P}ic_{X/S}(T) \tag{3.3}$$

for each $T \in Sch_{\mathbf{C}}/S$ is a monomorphism (and hence $Pic_{X/S}$ defines a separated presheaf on $Sch_{\mathbf{C}}/S$) or an isomorphism (i.e. $Pic_{X/S}$ defines a sheaf on $Sch_{\mathbf{C}}/S$). We start by defining locally free $\mathcal{O}_Y$-modules on a bicomplete scheme $(Y, \mathcal{O}_Y) \in Sch_{\mathbf{C}}$.

Definition 3.1. Let $(Y, \mathcal{O}_Y)$ be a bicomplete scheme and let $\mathcal{M} \in \mathcal{O}_Y - QCoh$. Let $m \in \mathbb{Z}$ be a non-negative integer. We will say that $\mathcal{M}$ is locally free of rank m if there exists an affine Zariski cover $(U_i \to Y)_{i \in I}$ of Y such that $\mathcal{M}(U_i) \cong \mathcal{O}_Y(U_i)^m$ for each $i \in I$. We will often refer to such an object $\mathcal{M}$ simply as locally free. In particular, if $m = 1$, we will say that $\mathcal{M}$ is invertible.

If Y is a bicomplete scheme and $\mathcal{M} \in \mathcal{O}_Y - QCoh$ is locally free, it is clear from condition (b) in Definition 2.3 that $\mathcal{M}$ actually lies in $\mathcal{O}_Y - Coh$. The full subcategory of $\mathcal{O}_Y - Coh$ consisting of locally free $\mathcal{O}_Y$-modules will be denoted by $\mathcal{P}(\mathcal{O}_Y - Coh)$. We also note that $\mathcal{M} \in \mathcal{O}_Y - Coh$ is locally free if and only if there exists a Zariski covering $(U_i \to Y)_{i \in I}$ of Y and a non-negative integer m such that on each restriction $\mathcal{M}|U_i$, $i \in I$, we have an isomorphism $\mathcal{M}|U_i \cong \mathcal{O}_{U_i}^m$.

Lemma 3.2. *Let $(Y, \mathcal{O}_Y)$ be a bicomplete scheme. Then*:

(a) *The locally free $\mathcal{O}_Y$-modules are dualizable, i.e. $\mathcal{P}(\mathcal{O}_Y - Coh)$ is a subcategory of $\mathscr{D}(\mathcal{O}_Y - Coh)$.*
(b) *If $\mathcal{M}$, $\mathcal{M}' \in \mathcal{P}(\mathcal{O}_Y - Coh)$, we have $\underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{M}') \in \mathcal{P}(\mathcal{O}_Y - Coh)$.*
(c) *If $(f, f^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is a morphism of bicomplete schemes in $Sch_{\mathbf{C}}$, for any $\mathcal{N}$, $\mathcal{N}' \in \mathcal{P}(\mathcal{O}_Z - Coh)$, we have*

$$f^* \underline{Hom}_{\mathcal{O}_Z}\big(\mathcal{N}, \mathcal{N}'\big) \cong \underline{Hom}_{\mathcal{O}_Y}\big(f^*\mathcal{N}, f^*\mathcal{N}'\big) \tag{3.4}$$

Proof. (a) Let $\mathcal{M} \in \mathcal{P}(\mathcal{O}_Y - Coh)$. We consider the natural morphism

$$\underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{M} \longrightarrow \underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{M}) \tag{3.5}$$

By Definition 3.1, there exists a Zariski affine cover $(U_i \to Y)_{i\in I}$ of Y and a non-negative integer m such that $\mathcal{M}(U_i) \cong \mathcal{O}_Y(U_i)^m$ for all $i \in I$. Hence, we have, for each $i \in I$:

$$\begin{aligned}\big(\underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{M}\big)(U_i) &\cong \underline{Hom}_{\mathcal{O}_Y(U_i)}\big(\mathcal{M}(U_i), \mathcal{O}_Y(U_i)\big) \otimes_{\mathcal{O}_Y(U_i)} \mathcal{M}(U_i)\\ &\cong \underline{Hom}_{\mathcal{O}_Y(U_i)}\big(\mathcal{O}_Y(U_i)^m, \mathcal{O}_Y(U_i)\big) \otimes_{\mathcal{O}_Y(U_i)} \mathcal{O}_Y(U_i)^m\\ &\cong \underline{Hom}_{\mathcal{O}_Y(U_i)}\big(\mathcal{O}_Y(U_i)^m, \mathcal{O}_Y(U_i)^m\big)\\ &\cong \underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{M})(U_i) \end{aligned} \tag{3.6}$$

Since the U_i form an affine cover of Y, it follows from (3.6) that the morphism in (3.5) is locally an isomorphism. Hence, we have $\underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{M} \cong \underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{M})$, i.e., $\mathcal{M}$ is dualizable.

(b) Since $\mathcal{M}, \mathcal{M}' \in \mathcal{P}(\mathcal{O}_Y - Coh)$, we can choose a Zariski affine cover $(U_i \to Y)_{i\in I}$ and non-negative integers m and m' such that $\mathcal{M}(U_i) \cong \mathcal{O}_Y(U_i)^m$ and $\mathcal{M}'(U_i) \cong \mathcal{O}_Y(U_i)^{m'}$ for all $i \in I$. Hence,

$$\underline{Hom}_{\mathcal{O}_Y}\big(\mathcal{M}, \mathcal{M}'\big)(U_i) \cong \underline{Hom}_{\mathcal{O}_Y(U_i)}\big(\mathcal{O}_Y(U_i)^m, \mathcal{O}_Y(U_i)^{m'}\big) \cong \mathcal{O}_Y(U_i)^{mm'} \tag{3.7}$$

from which it follows that $\underline{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{M}') \in \mathcal{P}(\mathcal{O}_Y - Coh)$.

(c) We consider the natural morphism

$$f^*\underline{Hom}_{\mathcal{O}_Z}\big(\mathcal{N}, \mathcal{N}'\big) \longrightarrow \underline{Hom}_{\mathcal{O}_Y}\big(f^*\mathcal{N}, f^*\mathcal{N}'\big) \tag{3.8}$$

that corresponds to the "evaluation morphism" given by the composition

$$f^*\mathcal{N} \otimes_{\mathcal{O}_Y} f^*\underline{Hom}_{\mathcal{O}_Z}\big(\mathcal{N}, \mathcal{N}'\big) \cong f^*\big(\mathcal{N} \otimes_{\mathcal{O}_Z} \underline{Hom}_{\mathcal{O}_Z}\big(\mathcal{N}, \mathcal{N}'\big)\big) \longrightarrow f^*\mathcal{N}' \tag{3.9}$$

using the isomorphism

$$\begin{aligned}&Hom_{\mathcal{O}_Y - Coh}\big(f^*\underline{Hom}_{\mathcal{O}_Z}\big(\mathcal{N}, \mathcal{N}'\big), \underline{Hom}_{\mathcal{O}_Y}\big(f^*\mathcal{N}, f^*\mathcal{N}'\big)\big)\\ &\quad\cong Hom_{\mathcal{O}_Y - Coh}\big(f^*\mathcal{N} \otimes_{\mathcal{O}_Y} f^*\underline{Hom}_{\mathcal{O}_Z}\big(\mathcal{N}, \mathcal{N}'\big), f^*\mathcal{N}'\big)\end{aligned} \tag{3.10}$$

Since $\mathcal{N}, \mathcal{N}' \in \mathcal{P}(\mathcal{O}_Z - Coh)$, there exists an affine cover $(V_i \to Z)_{i\in I}$ and non-negative integers n and n' such that $\mathcal{N}(V_i) \cong \mathcal{O}_Y(V_i)^n$ and $\mathcal{N}'(V_i) \cong \mathcal{O}_Y(V_i)^{n'}$ for all $i \in I$. We define each U_i by the cartesian square

$$\begin{array}{ccc} U_i & \longrightarrow & V_i \\ \Big\downarrow & & \Big\downarrow \\ Y & \longrightarrow & Z \end{array} \tag{3.11}$$

For some $i \in I$, we choose $W \in ZarAff(U_i)$. Then, we have

$$\begin{aligned}f^*\underline{Hom}_{\mathcal{O}_Z}\big(\mathcal{N}, \mathcal{N}'\big)(W) &\cong \underline{Hom}_{\mathcal{O}_Z(V_i)}\big(\mathcal{N}(V_i), \mathcal{N}'(V_i)\big) \otimes_{\mathcal{O}_Z(V_i)} \mathcal{O}_Y(W)\\ &\cong \underline{Hom}_{\mathcal{O}_Z(V_i)}\big(\mathcal{O}_Z(V_i)^n, \mathcal{O}_Z(V_i)^{n'}\big) \otimes_{\mathcal{O}_Z(V_i)} \mathcal{O}_Y(W)\\ &\cong \underline{Hom}_{\mathcal{O}_Y(W)}\big(\mathcal{O}_Y(W)^n, \mathcal{O}_Y(W)^{n'}\big)\\ &\cong \underline{Hom}_{\mathcal{O}_Y(W)}\big(f^*\mathcal{N}(W), f^*\mathcal{N}'(W)\big)\\ &\cong \underline{Hom}_{\mathcal{O}_Y}\big(f^*\mathcal{N}, f^*\mathcal{N}'\big)(W)\end{aligned} \tag{3.12}$$

It follows that the morphism $f^*\underline{Hom}_{\mathcal{O}_Z}(\mathcal{N}, \mathcal{N}') \to \underline{Hom}_{\mathcal{O}_Y}(f^*\mathcal{N}, f^*\mathcal{N}')$ in (3.8) is locally an isomorphism. Hence, we have an isomorphism $f^*\underline{Hom}_{\mathcal{O}_Z}(\mathcal{N}, \mathcal{N}') \cong \underline{Hom}_{\mathcal{O}_Y}(f^*\mathcal{N}, f^*\mathcal{N}')$ in $\mathcal{O}_Y - Coh$. $\square$

Proposition 3.3. *Let $(Y, \mathcal{O}_Y)$ be a bicomplete scheme and let $\mathcal{L} \in \mathcal{O}_Y - Coh$ be invertible. Then, we have a natural isomorphism*:

$$D\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{L} = \underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{L} \longrightarrow \mathcal{O}_Y \tag{3.13}$$

Proof. By Definition 3.1, the invertible coherent $\mathcal{O}_Y$-module $\mathcal{L}$ is also locally free. It follows from Lemma 3.2 that $\mathcal{L}$ is also dualizable. Hence, we have an isomorphism

$$D\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{L} = \underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \mathcal{L} \xrightarrow{\cong} \underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{L}) \tag{3.14}$$

Further, the identity morphism in $Hom_{\mathcal{O}_Y - Coh}(\mathcal{L}, \mathcal{L}) = Hom_{\mathcal{O}_Y - Coh}(\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y, \mathcal{L})$ corresponds to a morphism $\mathcal{O}_Y \to \underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{L})$ in $Hom_{\mathcal{O}_Y - Coh}(\mathcal{O}_Y, \underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{L}))$. Let $(U_i \to Y)_{i \in I}$ be an affine Zariski cover of Y such that $\mathcal{L}(U_i) \cong \mathcal{O}_Y(U_i)$ $\forall\, i \in I$. We note that for each $i \in I$, we have an isomorphism

$$\mathcal{O}_Y(U_i) \xrightarrow{\cong} \underline{Hom}_{\mathcal{O}_Y(U_i)}\big(\mathcal{O}_Y(U_i), \mathcal{O}_Y(U_i)\big) \xrightarrow{\cong} \underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{L})(U_i) \tag{3.15}$$

It follows that we have an isomorphism $\mathcal{O}_Y \xrightarrow{\cong} \underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{L})$. This proves the result. □

From the proof of Lemma 3.2(b), we know that if $\mathcal{L}$ is an invertible (and hence locally free) object of $\mathcal{O}_Y - Coh$, its dual $\underline{Hom}_{\mathcal{O}_Y}(\mathcal{L}, \mathcal{O}_Y)$ is also locally free of rank 1 and hence invertible. It follows that the collection of isomorphism classes of invertible coherent $\mathcal{O}_Y$-modules forms a group.

Definition 3.4. Let $(Y, \mathcal{O}_Y)$ be a bicomplete scheme in $Sch_{\mathbf{C}}$. The group $Pic(Y)$ of isomorphism classes of invertible coherent $\mathcal{O}_Y$-modules will be referred to as the Picard group of Y. Sometimes, the dual $D\mathcal{M}$ of an invertible object $\mathcal{M}$ in $\mathcal{O}_Y - Coh$ will also be denoted by $\mathcal{M}^{-1}$.

For any bicomplete scheme Y, the category $\mathcal{O}_Y - Coh$ is symmetric monoidal and hence the group $Pic(Y)$ of isomorphism classes of invertible coherent $\mathcal{O}_Y$-modules is an abelian group. Further, from Proposition 2.8 and Lemma 3.2(c), it follows that if $f : Y \to Z$ is a morphism of bicomplete schemes, f induces a natural morphism $f^* : Pic(Z) \to Pic(Y)$ of abelian groups.

Remark 3.5. We mention here that there are several similar notions of Picard groups in the context of symmetric monoidal categories appearing in the literature (see, for instance, [5,9,14]). For a more general discussion on invertible objects in a symmetric monoidal category, see [12].

As described in (3.1), we fix a morphism $h : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ in $Sch_{\mathbf{C}}$ with the properties mentioned at the beginning of this section. For any bicomplete S-scheme T, we consider the fibre product $h_T : X_T := X \times_S T \to T$. By assumption on $h : X \to S$, we know that the scheme X_T is bicomplete. We then define the relative Picard functor $Pic_{X/S} : (Sch_{\mathbf{C}}/S)^{op} \to \mathbf{Ab}$ by setting:

$$Pic_{X/S}(T) = Pic(X_T)/h_T^* Pic(T) \tag{3.16}$$

and let $\mathscr{P}ic_{X/S}$ be the associated sheaf on $Sch_{\mathbf{C}}/S$. We can now generalize [10, Lemma 2.7] to the setting of schemes over a symmetric monoidal category.

Lemma 3.6. *Let $h : X \to S$ be a morphism of bicomplete schemes and suppose that the natural morphism $\mathcal{O}_S \to h_*\mathcal{O}_X$ is an isomorphism. Then, the functor $\mathcal{N} \mapsto h^*\mathcal{N}$ is fully faithful on the* *category $\mathcal{P}(\mathcal{O}_S - Coh)$. Further, the essential image of this functor is formed by objects $\mathcal{M}$ of $\mathcal{P}(\mathcal{O}_X - Coh)$ such that*

(1) *$h_*\mathcal{M}$ lies in $\mathcal{P}(\mathcal{O}_S - Coh)$, and*
(2) *The natural map $h^*h_*\mathcal{M} \to \mathcal{M}$ in $\mathcal{O}_X - Coh$ is an isomorphism.*

Proof. For any $\mathcal{N} \in \mathcal{P}(\mathcal{O}_S - Coh)$, we have isomorphisms

$$\mathcal{N} \cong \mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{O}_S \cong \mathcal{N} \otimes_{\mathcal{O}_S} h_*\mathcal{O}_X \cong h_*\big(h^*\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_X\big) \cong h_*h^*\mathcal{N} \tag{3.17}$$

where the isomorphism $\mathcal{N} \otimes_{\mathcal{O}_S} h_*\mathcal{O}_X \cong h_*(h^*\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_X)$ in (3.17) follows from Proposition 2.11 and Lemma 3.2(a). Given $\mathcal{N}', \mathcal{N}'' \in \mathcal{P}(\mathcal{O}_S - Coh)$, it follows from Lemma 3.2(b) that $\underline{Hom}_{\mathcal{O}_S}(\mathcal{N}', \mathcal{N}'') \in \mathcal{P}(\mathcal{O}_S - Coh)$. Hence, from (3.17), it follows that

$$\underline{Hom}_{\mathcal{O}_S}\big(\mathcal{N}', \mathcal{N}''\big) \cong h_*h^*\underline{Hom}_{\mathcal{O}_S}\big(\mathcal{N}', \mathcal{N}''\big) \tag{3.18}$$

Again, from Lemma 3.2(c), we know that $h^*\underline{Hom}_{\mathcal{O}_S}(\mathcal{N}', \mathcal{N}'') \cong \underline{Hom}_{\mathcal{O}_X}(h^*\mathcal{N}', h^*\mathcal{N}'')$. Combining with (3.18), we have

$$\underline{Hom}_{\mathcal{O}_S}\big(\mathcal{N}', \mathcal{N}''\big) \cong h_*\underline{Hom}_{\mathcal{O}_X}\big(h^*\mathcal{N}', h^*\mathcal{N}''\big) \tag{3.19}$$

It follows that

$$\begin{aligned} Hom_{\mathcal{O}_S-Coh}\big(\mathcal{N}', \mathcal{N}''\big) &\cong Hom_{\mathcal{O}_S-Coh}\big(\mathcal{O}_S, \underline{Hom}_{\mathcal{O}_S}\big(\mathcal{N}', \mathcal{N}''\big)\big) \\ &\cong Hom_{\mathcal{O}_S-Coh}\big(\mathcal{O}_S, h_*\underline{Hom}_{\mathcal{O}_X}\big(h^*\mathcal{N}', h^*\mathcal{N}''\big)\big) \\ &\cong Hom_{\mathcal{O}_X-Coh}\big(h^*\mathcal{O}_S, \underline{Hom}_{\mathcal{O}_X}\big(h^*\mathcal{N}', h^*\mathcal{N}''\big)\big) \\ &\cong Hom_{\mathcal{O}_X-Coh}\big(h^*\mathcal{N}', h^*\mathcal{N}''\big) \end{aligned} \tag{3.20}$$

Hence the functor $\mathcal{N} \mapsto h^*\mathcal{N}$ on the category $\mathcal{P}(\mathcal{O}_S - Coh)$ is full.

Now, suppose that $\mathcal{M} \in \mathcal{P}(\mathcal{O}_X - Coh)$ satisfies conditions (1) and (2). Then, since $h_*\mathcal{M} \in \mathcal{P}(\mathcal{O}_S - Coh)$ and $\mathcal{M} \cong h^*(h_*\mathcal{M})$, it follows directly that $\mathcal{M}$ is in the essential image of the functor h^* restricted to $\mathcal{P}(\mathcal{O}_S - Coh)$.

Conversely, suppose that $\mathcal{M} \in \mathcal{P}(\mathcal{O}_X - Coh)$ is in the essential image, i.e., there exists $\mathcal{N} \in \mathcal{P}(\mathcal{O}_S - Coh)$ such that $\mathcal{M} \cong h^*\mathcal{N}$. Then, using (3.17), we get

$$h_*\mathcal{M} \cong h_*h^*\mathcal{N} \cong \mathcal{N} \in \mathcal{P}(\mathcal{O}_S - Coh) \tag{3.21}$$

which proves (1). Further, applying h^* to (3.21), we get $h^*h_*\mathcal{M} \cong h^*\mathcal{N} \cong \mathcal{M}$. This proves (2). $\square$

Let $\{p_i\colon Y_i = Spec(A_i) \to Y = Spec(A)\}_{i\in I}$ be a Zariski affine covering of an affine scheme $Y = Spec(A)$ over $\mathbf{C}$. Then, by definition (see [13, Définition 2.10]), the collection of functors

$$\_\_ \otimes_A A_i : A - Mod \longrightarrow A_i - Mod \quad i \in I \tag{3.22}$$

is conservative, i.e., a morphism $f : M \to N$ in $A - Mod$ is an isomorphism if and only if each induced morphism $f \otimes_A A_i : M \otimes_A A_i \to N \otimes_A A_i$ is an isomorphism. From the definition of the pullback functors in Proposition 2.8, it follows that for any scheme $Y \in Sch_{\mathbf{C}}$, given a Zariski cover $\{p_i : Y_i \to Y\}_{i\in I}$ of Y, a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathcal{O}_Y - QCoh$ is an isomorphism if and only if each induced morphism

$$p_i^*(f) : p_i^*(\mathcal{M}) \longrightarrow p_i^*(\mathcal{N}) \quad i \in I \tag{3.23}$$

is an isomorphism. We set $Y' = \coprod_{i\in I} Y_i$ and consider $p : Y' \to Y$. Then, it follows that a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathcal{O}_Y - QCoh$ is an isomorphism if and only if

$$p^*(f) : p^*(\mathcal{M}) \longrightarrow p^*(\mathcal{N}) \tag{3.24}$$

is an isomorphism. We will refer to $p : Y' \to Y$ as a Zariski cover of Y. From the construction, it is clear that the property of being a Zariski cover is stable under base change. We also know that each of the functors $\_\_ \otimes_A A_i : A - Mod \to A_i = Mod$ in (3.22) commutes with finite limits (because each A_i is a flat A-module). Hence, given a Zariski cover $p : Y' \to Y$, the pullback functor $p^* : \mathcal{O}_Y - QCoh \to \mathcal{O}_{Y'} - QCoh$ commutes with finite limits. It is also clear that $\mathcal{M} \in \mathcal{O}_Y - QCoh$ is invertible (i.e., locally free of rank 1) if and only if $p^*\mathcal{M}$ is invertible.

We are now ready to prove part (a) of Theorem 1.1 mentioned in the introduction. We recall here that for the given bicomplete scheme S as in (3.1), $Sch_{\mathbf{C}}/S$ always denotes the category of bicomplete schemes over S.

Proposition 3.7. *Let $h : X \to S$ be a morphism of bicomplete schemes as in* (3.1) *and suppose that the natural morphism $\mathcal{O}_S \to h_*\mathcal{O}_X$ is universally an isomorphism, i.e., for any bicomplete S-scheme T, the natural morphism $\mathcal{O}_T \to h_{T*}\mathcal{O}_{X_T}$ is an isomorphism $\mathcal{O}_T \xrightarrow{\cong} h_{T*}\mathcal{O}_{X_T}$, where X_T denotes the fibre product $X_T := X \times_S T$.*

Then, for any object $T \in Sch_{\mathbf{C}}/S$, we have an injection

$$Pic_{X/S}(T) \hookrightarrow \mathscr{P}ic_{X/S}(T) \tag{3.25}$$

*of abelian groups. Consequently, the functor $Pic_{X/S} : (Sch_{\mathbf{C}}/S)^{op} \to$ **Ab** defines a separated presheaf on $Sch_{\mathbf{C}}/S$.*

Proof. Suppose that $\lambda \in Pic_{X/S}(T)$ maps to 0 in $\mathscr{P}ic_{X/S}(T)$. Let λ be represented by some invertible locally free $\mathcal{L} \in \mathcal{P}(\mathcal{O}_{X_T} - Coh)$. It follows that there exists a Zariski cover $p : T' \to T$ of T such that $p_X^*\mathcal{L} \cong h_{T'}^*\mathcal{N}'$ for some locally free invertible $\mathcal{N}' \in \mathcal{P}(\mathcal{O}_{T'} - Coh)$; the notation being as in the following diagram in which all squares are cartesian:

$$\begin{tikzcd} X_{T'} \arrow[r, "h_{T'}"] \arrow[d, "p_X"'] & T' \arrow[d, "p"] \\ X_T \arrow[r, "h_T"] \arrow[d] & T \arrow[d] \\ X \arrow[r, "h"] & S \end{tikzcd} \tag{3.26}$$

Using condition (1), we have $\mathcal{O}_{T'} \cong h_{T'*}\mathcal{O}_{X_T}$. Then, it follows from (3.17) in the proof of Lemma 3.6 that

$$h_{T'*}p_X^*\mathcal{L} \cong h_{T'*}h_{T'}^*\mathcal{N}' \cong \mathcal{N}' \tag{3.27}$$

Since p and p_X are Zariski covers, the functors p^* and p_X^* commute with finite limits. Hence, from the description of the pushforward functors in (2.24), we have an isomorphism $p^*h_{T*} \cong h_{T'*}p_X^*$ of functors. Combining this with (3.27), it follows that we have an isomorphism

$$p^*h_{T*}\mathcal{L} \cong h_{T'*}p_X^*\mathcal{L} \cong \mathcal{N}' \tag{3.28}$$

Since $p^*h_{T*}\mathcal{L} = \mathcal{N}'$ is invertible and $p : T' \to T$ is a Zariski cover of T, it follows that $h_{T*}\mathcal{L}$ is invertible and defines a class in $Pic(T)$.

Further, since $p_X^*\mathcal{L} \cong h_{T'}^*\mathcal{N}'$ lies in the essential image of $h_{T'}^*$ and $\mathcal{N}' \in \mathcal{P}(\mathcal{O}_{T'} - Coh)$, it follows from Lemma 3.6 that $h_{T'}^*h_{T'*}p_X^*\mathcal{L} \cong p_X^*\mathcal{L}$. Combining with (3.28), we have

$$p_X^*h_T^*h_{T*}\mathcal{L} \cong h_{T'}^*p^*h_{T*}\mathcal{L} \cong h_{T'}^*h_{T'*}p_X^*\mathcal{L} \cong p_X^*\mathcal{L} \tag{3.29}$$

Since $p_X : X_{T'} \to X_T$ is also a Zariski cover, it follows from (3.29) that $h_T^*h_{T*}\mathcal{L} \cong \mathcal{L}$. It is now clear that the class $\lambda \in Pic_{X/S}(T) = Pic(X_T)/h_T^*Pic(T)$ represented by $\mathcal{L}$ is zero. $\square$

We will now give sufficient conditions for $\mathscr{P}ic_{X/S}$ to be a sheaf on $Sch_{\mathbf{C}}/S$. Suppose that $h : X \to S$ has a section g, so that $hg = 1$. Then, for any bicomplete $T \in Sch_{\mathbf{C}}/S$, the morphism $h_T : X_T := X \times_S T \to T$ has a section, which we denote by $g_T : T \to X_T$. Given an invertible, locally free object $\mathcal{L} \in \mathcal{P}(\mathcal{O}_{X_T} - Coh)$, by analogy with [10, Definition 2.8], we will say that a g-rigidification of $\mathcal{L}$ is a chosen isomorphism $u : \mathcal{O}_T \xrightarrow{\cong} g_T^*\mathcal{L}$, if it exists. By an isomorphism $(\mathcal{L}, u) \cong (\mathcal{L}', u')$ of pairs, we mean an isomorphism $v : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ such that the composition $g_T^*(v) \circ u : \mathcal{O}_T \xrightarrow{\cong} g_T^*\mathcal{L} \xrightarrow{\cong} g_T^*\mathcal{L}'$ equals $u' : \mathcal{O}_T \xrightarrow{\cong} g_T^*\mathcal{L}'$. Given the section $g : S \to X$ and $T \in Sch_{\mathbf{C}}/S$, we consider the group $R_g(T)$ of isomorphism classes of such pairs $(\mathcal{L}, u)$. Then, the following lemma describes the group $R_g(T)$ in terms of the relative Picard group $Pic_{X/S}(T)$ and also generalizes [10, Lemmas 2.9–2.10].

Lemma 3.8. *Suppose that $h : X \to S$ has a section g and suppose that the natural morphism $\mathcal{O}_S \to h_*\mathcal{O}_X$ is universally an isomorphism, i.e., for any bicomplete S-scheme T, the natural morphism $\mathcal{O}_T \to h_{T*}\mathcal{O}_{X_T}$ is an isomorphism $\mathcal{O}_T \xrightarrow{\cong} h_{T*}\mathcal{O}_{X_T}$, where X_T denotes the fibre product $X_T := X \times_S T$. Let $T \in Sch_{\mathbf{C}}/S$. Then, we have*:

(a) *The group $R_g(T)$ of isomorphism classes of pairs $(\mathcal{L}, u)$ is isomorphic to $Pic_{X/S}(T)$ by means of the map $\rho(\mathcal{L}, u) = \mathcal{L}$.*
(b) *For any such pair $(\mathcal{L}, u) \in R_g(T)$, the set of automorphisms of $(\mathcal{L}, u)$ is trivial.*

Proof. (a) It is clear that the pair $(\mathcal{O}_{X_T}, 1)$ is the identity element of the group $R_g(T)$. Let $\lambda \in Pic_{X/S}(T)$ be represented by some locally free invertible $\mathcal{M} \in \mathcal{O}_{X_T} - Coh$. Then, $\lambda \in Pic_{X/S}(T) = Pic(X_T)/h_T^*Pic(T)$ is also represented by $\mathcal{L} = \mathcal{M} \otimes h_T^*g_T^*(\mathcal{M})^{-1}$. Then, using Proposition 2.8, the following sequence of isomorphisms:

$$g_T^*\mathcal{L} \cong g_T^*\mathcal{M} \otimes g_T^*h_T^*g_T^*(\mathcal{M})^{-1} = g_T^*\mathcal{M} \otimes g_T^*(\mathcal{M})^{-1} \cong g_T^*\mathcal{O}_{X_T} \cong \mathcal{O}_T \tag{3.30}$$

defines a g-rigidification of $\mathcal{L}$. Hence, ρ is surjective.

On the other hand, suppose that the pair $(\mathcal{L}, u)$ lies in the kernel of ρ. By definition, it follows that there exists some locally free invertible $\mathcal{N} \in \mathcal{O}_T - Coh$ and an isomorphism $v : \mathcal{L} \xrightarrow{\cong} h_T^*\mathcal{N}$. We set $w := g_T^*(v) \circ u : \mathcal{O}_T \xrightarrow{\cong} g_T^*\mathcal{L} \xrightarrow{\cong} g_T^*h_T^*\mathcal{N} \cong \mathcal{N}$. Then, we have isomorphisms of pairs

$$v : (\mathcal{L}, u) \xrightarrow{\cong} \big(h_T^*\mathcal{N}, w\big) \qquad h_T^*(w) : (\mathcal{O}_{X_T}, 1) \xrightarrow{\cong} \big(h_T^*\mathcal{N}, w\big) \tag{3.31}$$

from which it follows that $(\mathcal{L}, u)$ is trivial. Hence, ρ is an isomorphism.

(b) Suppose that $v : \mathcal{L} \xrightarrow{\cong} \mathcal{L}$ is an automorphism of the pair $(\mathcal{L}, u)$, i.e. $g_T^*(v) \circ u = u$. Since u is an isomorphism, we have $g_T^*(v) = 1$. From Lemma 3.2(a), we know that any locally free object in $\mathcal{O}_{X_T} - Coh$ is dualizable. Hence, given an invertible, locally free object $\mathcal{L}$ in $\mathcal{O}_{X_T} - Coh$, we have

$$\mathcal{O}_{X_T} \cong \underline{Hom}_{\mathcal{O}_{X_T}}(\mathcal{L}, \mathcal{O}_{X_T}) \otimes_{\mathcal{O}_{X_T}} \mathcal{L} \cong D\mathcal{L} \otimes_{\mathcal{O}_{X_T}} \mathcal{L} \cong \underline{Hom}_{\mathcal{O}_{X_T}}(\mathcal{L}, \mathcal{L}) \tag{3.32}$$

Then, we get

$$\begin{aligned} v \in Hom_{\mathcal{O}_{X_T}-Coh}(\mathcal{L}, \mathcal{L}) &\cong Hom_{\mathcal{O}_{X_T}-Coh}\big(\mathcal{O}_{X_T}, \underline{Hom}_{\mathcal{O}_{X_T}}(\mathcal{L}, \mathcal{L})\big) \\ &\cong Hom_{\mathcal{O}_{X_T}-Coh}(\mathcal{O}_{X_T}, \mathcal{O}_{X_T}) \end{aligned} \tag{3.33}$$

Similarly, $g_T^*(v) \in Hom_{\mathcal{O}_T-Coh}(g_T^*\mathcal{L}, g_T^*\mathcal{L}) \cong Hom_{\mathcal{O}_T-Coh}(\mathcal{O}_T, \mathcal{O}_T)$. Further, from the fact that $h_T^*\mathcal{O}_T \cong \mathcal{O}_{X_T}$ and from (3.20) in the proof of Lemma 3.6, it follows that

$$Hom_{\mathcal{O}_{X_T}-Coh}(\mathcal{O}_{X_T}, \mathcal{O}_{X_T}) \cong Hom_{\mathcal{O}_T-Coh}(\mathcal{O}_T, \mathcal{O}_T) \tag{3.34}$$

Now, since $g_T^*(v) = 1 \in Hom_{\mathcal{O}_T-Coh}(g_T^*\mathcal{L}, g_T^*\mathcal{L}) = Hom_{\mathcal{O}_T-Coh}(\mathcal{O}_T, \mathcal{O}_T)$, it follows from (3.33) and (3.34) that $v = 1$. □

Proposition 3.9. *Let $h : X \to S$ be a morphism of bicomplete schemes as in* (3.1) *that further has the following two properties*:

(1) *The natural morphism $\mathcal{O}_S \to h_*\mathcal{O}_X$ is universally an isomorphism, i.e., for any bicomplete S-scheme T, the natural morphism $\mathcal{O}_T \to h_{T*}\mathcal{O}_{X_T}$ is an isomorphism $\mathcal{O}_T \xrightarrow{\cong} h_{T*}\mathcal{O}_{X_T}$, where X_T denotes the fibre product $X_T := X \times_S T$.*
(2) *The morphism $h : X \to S$ has a section $g : S \to X$ so that $hg = 1$.*

Then, for any $T \in Sch_{\mathbf{C}}/S$, the natural morphism

$$Pic_{X/S}(T) \longrightarrow \mathscr{P}ic_{X/S}(T) \tag{3.35}$$

is an isomorphism, i.e., the relative Picard functor $Pic_{X/S}$ determines a sheaf on $Sch_{\mathbf{C}}/S$.

Proof. From Proposition 3.7, we know that for any $T \in Sch_{\mathbf{C}}/S$, the natural morphism $Pic_{X/S}(T) \to \mathscr{P}ic_{X/S}(T)$ is an injection and hence $Pic_{X/S}$ determines a separated presheaf on $Sch_{\mathbf{C}}/S$. We consider any $\lambda \in \mathscr{P}ic_{X/S}(T)$, corresponding to some $\lambda' \in Pic_{X/S}(T')$, where $T' \to T$ is a Zariski cover of T. Since $\mathscr{P}ic_{X/S}$ is a sheaf, it follows that there exists a Zariski cover $T'' \to T' \times_T T'$ such that the two pullbacks of λ' to $X_{T''}$ are identical. Since $Pic_{X/S}$ is a separated presheaf, we may let $T'' \cong T' \times_T T'$.

Using Lemma 3.8, we may suppose that $\lambda' \in Pic_{X/S}(T')$ is represented by a pair $(\mathcal{L}', u') \in R_g(T')$, u' being a g-rigidification of a locally free invertible object $\mathcal{L}' \in \mathcal{O}_{X_{T'}} - Coh$. Let $p_1 : X_{T''} \to X_{T'}$ and $p_2 : X_{T''} \to X_{T'}$ denote the two coordinate projections. Since $p_1^*(\lambda') = p_2^*(\lambda') \in Pic_{X/S}(T'')$, it follows from Lemma 3.8 that there exists an isomorphism $v'' : p_1^*(\mathcal{L}') \xrightarrow{\cong} p_2^*(\mathcal{L}')$.

We consider $T''' := T' \times_T T' \times_T T'$ and let v_{ij}''' denote the pullback of the isomorphism v'' to T''' via the projection $p_{ij} : T''' \to T' \times_T T'$ on the i-th and j-th factors. Then, ${v'''}_{13}^{-1} v_{23}''' v_{12}'''$ is an automorphism of the pair $(p_{12}^* p_1^* \mathcal{L}', p_{12}^* p_1^* u') \in R_g(T''')$. Again, using Lemma 3.8, it follows that this automorphism is trivial, i.e., we have $v_{13}''' = v_{23}''' v_{12}'''$.

From the above, it follows that we have obtained a descent datum on X_T, which may be pulled back to a descent datum on any object $U \to X_T$ in $ZarAff(X_T)$. Using [13, Théorème 2.5], [13, Corollaire 2.11] and the fact that X_T is bicomplete, it follows that the descent datum on each such object $U \to X_T$ in $ZarAff(X_T)$ determines an object $\mathcal{L}_U \in \mathcal{O}_{X_T}(U) - Coh$. Then, the functor

$$\mathcal{L} : ZarAff(X_T)^{op} \longrightarrow Mod_{\mathbf{C}} \qquad \mathcal{L}(U) := \big(\mathcal{O}_{X_T}(U), \mathcal{L}_U\big) \tag{3.36}$$

determines an object $\mathcal{L} \in \mathcal{O}_{X_T} - Coh$ restricting to the invertible $\mathcal{O}_{X_{T'}}$-module $\mathcal{L}'$. Since $X_{T'} \to X_T$ is a Zariski cover of X_T, it follows that $\mathcal{L}$ is invertible. Hence, $\lambda \in Pic_{X/S}(T)$. □

Proposition 3.7 and Proposition 3.9 together complete the proof of Theorem 1.1 mentioned in Section 1.

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